CENTRAL LIMIT THEOREMS FOR
THE ERGODIC ADDING MACHINE

BY

Mordechay B. Levin and Ely Merzbach

Department of Mathematics and Computer Science
Bar-Ilan University, Ramat-Gan 52900, Israel
e-mail: mlevin@macs.biu.ac.il, merzbach@macs.biu.ac.il

ABSTRACT

In this paper we find a second class of sequences of random numbers $(x_n)_{n=1}^{\infty}$ (the orbit of the ergodic adding machine) such that the corresponding sequences of zeros and ones $1_{[0,y)}(x_n)$ $(n = 1, 2, \ldots, N)$ satisfy Central Limit Theorems with extremely small standard deviation $\sigma_N = O(\sqrt{\log N})$, instead of $O(\sqrt{N})$, as $N \to \infty$.

Dedicated to Professor Benjamin Weiss on the occasion of his 60th birthday.

1. Introduction

Let $(\beta_n)_{n=0}^{\infty}$ be a sequence of real numbers from the unit interval $[0, 1)$, $1_{[0,y)}(x)$ be the indicator function of the interval $[0, y)$:

$$(1.1) \qquad 1_{[0,y)}(x) = \begin{cases} 1, & \text{if } x \in [0, y) \\ 0, & \text{otherwise}; \end{cases}$$

$\{v\}$ is the fractional part of v; $[v]$ is the integer part of v, i.e., $v = [v] + \{v\}$; $\llcorner v \lrcorner = v - 1$ for the integer v, and $\llcorner v \lrcorner = [v]$, otherwise.

According to Roth's theorem (see Appendix)

$$(1.2) \qquad \inf_{\rho \in \mathbb{R}} \int_0^1 \int_0^1 \left(\sum_{n=0}^{\llcorner zN \lrcorner} (1_{[0,y)}(\beta_n) - y) - \rho \right)^2 dy\,dz \geq 2^{-8} \log_2 N.$$

Received May 24, 2001

Let $z, y \in [0,1)$ be uniformly distributed random variables. We see that the standard deviation of the random variable $\sum_{n=0}^{\llcorner zN \lrcorner}(1_{[0,y)}(\beta_n) - y)$ is bigger than $\sqrt{2^{-8}\log_2 N}$ for the entire random or deterministic sequence $(\beta_n)_{n \geq 0}$.

In [B1], [B2], [B3], Beck found some results on the stochastic behavior of the rotation on the circle. In particular, he proved

$$(1.3) \qquad \text{volume} \left\{ (u,y,z) \in [0,1)^3 \, \middle| \, \frac{\sum_{n=0}^{\llcorner zN \lrcorner}(1_{[u,u+y)}(\alpha n) - y)}{c_1 \sqrt{\log N}} < t \right\} \to \Phi(t)$$

uniformly for all t as $N \to \infty$, where α is a quadratic irrational number,

$$(1.4) \qquad\qquad\qquad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

and

$$(1.5) \qquad\qquad\qquad 1_{[u,u+y)}(v) = 1_{[0,y)}(\{v - u\}).$$

According to Roth's theorem, this is a first class of sequences $(\beta_n)_{n=0}^{\infty}$ ($\beta_n = \alpha n$, $n = 0, 1, \ldots$) such that the triangular array with random variable $1_{[u,u+y)}(\beta_n) - y$ satisfy the Central Limit Theorem (CLT) with extremely small (by order of magnitude) standard deviation.

In this paper we find a second class of such sequences, namely,

$$\beta_n = T_q^n(x), \quad n = 0, 1, 2, \ldots,$$

where $q \geq 2$ is an integer, and $T_q(x)$ is the von Neumann–Kakutani's ergodic adding machine: let $x = .x_1 x_2 \ldots x_k$ and $x' = .x_1' x_2' \ldots x_k'$ be the q-expansion of numbers x and $x' \in [0,1)$, $T_q(x) = x'$ is defined by

$$(1.6) \qquad\qquad x_k' = \begin{cases} 0, & \text{if } k = 1, 2, \ldots, i-1, \\ x_i + 1, & \text{if } k = i, \\ x_i, & \text{otherwise,} \end{cases}$$

where $x_k = q - 1$ for $k = 1, 2, \ldots, i-1$ and $x_i \neq q - 1$;

$$T_q^n(x) = T_q(T_q^{n-1}(x)), \quad n = 2, 3, \ldots, \quad T^0(x) = x.$$

The detailed description of the ergodic adding machine is given in [Fr, pp. 75–83] and in [Pe, pp. 208–212]. As is known, the sequence $(T_q^n(x))_{n \geq 1}$ coincides for $x = 0$ with the van der Corput sequence (see for example [LP], [P]).

We will prove the following three theorems:

THEOREM 1.1: *There exist constants* $c_1, c_2 > 0$ *such that*

$$\left| volume\left\{ (y,z) \in [0,1)^2 \left| \frac{\sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y) - e(x,u,N))}{\sqrt{\sigma_1(x,u,N)\log_q N}} < t \right. \right\} - \Phi(t) \right|$$

(1.7) $$\leq c_1(\log_q N)^{-1/7},$$

where

(1.8) $$e(x,u,N) = \int_0^1 \int_0^1 \sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y)dydz$$

and
(1.9)
$$\sigma_1(x,u,N) = \int_0^1 \int_0^1 \left(\sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y) - e(x,u,N) \right)^2 dydz/\log_q N$$
$$\in [2^{-12}\log_2 q, 2^{12}q^4]$$

for all $(u,x) \in [0,1)^2$, $t \in \mathbb{R}$ *and* $N \geq c_2$.

THEOREM 1.2: *There exist constants* $c_3, c_4 > 0$ *such that*

$$\left| volume\left\{ (x,y,z) \in [0,1)^3 \left| \frac{\sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y)}{\sqrt{\sigma_2(u,N)\log_q N}} < t \right. \right\} - \Phi(t) \right|$$

(1.10) $$\leq c_3(\log_q N)^{-1/7},$$

where

(1.11) $$\sigma_2(u,N) = \int_{[0,1)^3} \left(\sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y) \right)^2 dxdydz/\log_q N$$
$$\in [2^{-12}\log_2 q, 2^{12}q^4]$$

for all $u \in [0,1)$, $t \in \mathbb{R}$, *and* $N \geq c_4$.

THEOREM 1.3: *There exist constants* $c_5, c_6 > 0$ *such that*

$$\left| volume\left\{ (u,y,z) \in [0,1)^3 \left| \frac{\sum_{n=0}^{\lfloor zN \rfloor}(1_{[u,u+y)}(T_q^n(x)) - y)}{\sqrt{\sigma_3(x,N)\log_q N}} < t \right. \right\} - \Phi(t) \right|$$

(1.12) $$< c_5(\log_q N)^{-1/7},$$

where

$$
(1.13) \quad \sigma_3(x, N) = \int_{[0,1)^3} \left(\sum_{n=0}^{\lfloor zN \rfloor} (1_{[u,u+y)}(T_q^n(x)) - y) \right)^2 dudydz/\log_q N
$$

$$
\in [2^{-12} \log_2 q, 2^{12} q^4]
$$

for all $x \in [0, 1)$, $t \in \mathbb{R}$, *and* $N \geq c_6$.

The true error term in Theorem 1.1–Theorem 1.3 is $O((\log_q N)^{-1/6-\epsilon})$; for simplicity we write $O((\log_q N)^{-1/7})$.

Remark 1.1: It is easy to prove that $\sigma_1(x, u, N)$, $\sigma_2(u, N)$ and $\sigma_3(x, N)$ do not depend on u, x, and N for $x = m_1/q^r$, $u = m_2/q^r$ with integers m_1, m_2 and r; and for almost all $(u, x) \in [0, 1)^2$.

Remark 1.2: Let $q_n \geq 2$ $(n = 1, 2, \ldots)$ be a sequence of integers. Consider Cantor's expansion of $x \in [0, 1)$:

$$
x = .x_1 x_2 \ldots = \sum_{n=1}^{\infty} \frac{x_n}{q_1 \cdots q_n}, \quad \text{with } x_n \in \{0, 1, \ldots, q_n - 1\}, n = 1, 2 \ldots
$$

and the odometer transform $T(x) = x' = .x_1' x_2' \ldots$ where

$$
x_k' = \begin{cases} 0, & \text{if } k = 1, 2, \ldots, i - 1 \\ x_i + 1, & \text{if } k = i \\ x_i, & \text{otherwise} \end{cases}
$$

with $x_k = q_k - 1$ for $k = 1, 2, \ldots, i - 1$ and $x_i \neq q_i - 1$.

Repeating the proof of Theorem 1.1–Theorem 1.3, we obtain that they are true for the case of $\sum_{i=1}^{n} q_i = O(n)$, as $n \to \infty$. The case of the arbitrary sequence $(q_n)_{n \geq 1}$ will be considered in a forthcoming paper. We note that $(T^n(0))_{n \geq 1}$ coincides with the generalized van der Corput sequence [PA].

Remark 1.3: In forthcoming papers we will prove that the considered triangular array of random variables $(1_{[u,u+y)}(T_q^n(x)) - y)_{n=0}^{\lfloor zN \rfloor}$ satisfies the law of the iterated logarithm, almost sure CLT, the invariance principle, and the large deviation theorem.

We will also prove that CLT is valid for the generalized von Neumann–Kakutani's transform in the sense of [LP], [P], and for the following 1- and 2-parameter random sequences:

$$
f_1(x, N) = \#\{k \in [1, N] : T_q^k(x) < c/k\}, \quad x \in [0, 1), N = 1, 2, \ldots
$$

and

$$f_2(x, z, N) = \sum_{k=1}^{\lfloor zN \rfloor} T_q^k(x) - \lfloor zN \rfloor/2, \quad x, z \in [0,1)^2, \quad N = 1, 2, \ldots$$

(compare with theorems from [B1], [B2], [B3]).

Theorem 1.3 was announced in [L].

2. Auxiliary lemmas

We will use the following notation:

$$T(x) = T_q(\{x\}),$$

where $\{x\}$ is the fractional part of x;

$$(2.1) \qquad\qquad I(x < y) = \begin{cases} 1, & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we define functions $I(x \leq y)$, $I(x > y)$, and $I(x \geq y)$.

Let $[x] + .x_1 x_2 \ldots$ be the q-expansion of x, w, $x_i \in \{0, 1, \ldots, q-1\}$, $i = 1, 2, \ldots$. We denote

$$(2.2) \qquad\qquad R_k(x) = .x_k x_{k-1} \ldots x_1 = \sum_{i=1}^{k} x_i q^{i-k-1},$$

$$(2.3) \qquad\qquad \{x\}_k = .x_1 \ldots x_k = [q^k \{x\}]/q^k \quad \text{for } k = 1, 2, \ldots.$$

LEMMA 2.1: *Let $k, a, b \geq 0$ be integers, and $x \in [0, 1)$. Then*

$$(2.4) \qquad\qquad T^{aq^k + b}(x) = T^b \Big(\frac{[q^k x] + T^a(q^k x)}{q^k} \Big).$$

Proof: For $a = 0$ the equality (2.4) is clear. Let (2.4) be true for $a \geq 0$ and for all integers $k, b \geq 0$. Then

$$T^{(a+1)q^k + b}(x) = T^b(T^{aq^k + q^k}(x)) = T^b(v)$$

with

$$v = T^{q^k} \Big(\frac{[q^k x] + T^a(q^k x)}{q^k} \Big).$$

Now from (1.6), we obtain

$$v = ([q^k x] + T^{a+1}(q^k x))/q^k.$$

Lemma 2.1 is now proved by induction.   ∎

LEMMA 2.2: *Let* $x, y \in [0,1)$, $i \geq 1$ *be an integer. Then*

$$(2.5) \quad \sum_{k=0}^{q^i-1} (1_{[0,y)}(T^k(x)) - y) = - \{q^i y\} + I(R_i(x) \leq R_i(y))I(\{q^i x\} < \{q^i y\})$$

$$+ I(R_i(x) > R_i(y))I(T(q^i x) < \{q^i y\}).$$

Proof: The sequence $\{T^k(x)\}_i$ ($k = 0, 1, \ldots, q^i - 1$) passes the set

$$\{0, 1/q^i, 2/q^i, \ldots, (q^i - 1)/q^i\}.$$

Hence there exists an integer $k_0 \in \{0, 1, \ldots, q^i - 1\}$ such that

$$(2.6) \quad \{T^{k_0}(x)\}_i = \{y\}_i.$$

It is easy to see that

$$(2.7) \quad \sum_{k=0}^{q^i-1} 1_{[0,\{y\}_i)}(T^k(x)) = q^i\{y\}_i = q^i y - \{q^i y\}.$$

By (2.6) we have

$$(2.8) \quad \begin{aligned} \sum_{k=0}^{q^i-1} 1_{[\{y\}_i, y)}(T^k(x)) &= 1_{[\{y\}_i, y)}(T^{k_0}(y)) \\ &= 1_{[0, \{q^i y\}/q^i)}(\{T^{k_0}(x) - \{y\}_i\}) \\ &= 1_{[0, \{q^i y\}/q^i)}(\{q^i T^{k_0}(x)\}/q^i) \\ &= I(\{q^i T^{k_0}(x)\} < \{q^i y\}). \end{aligned}$$

We denote the left side of (2.5) by σ. From (2.7) and (2.8), we conclude that

$$(2.9) \quad \sigma = -\{q^i y\} + I(\{q^i T^{k_0}(x)\} < \{q^i y\}).$$

Let now

$$(2.10) \quad k_1 = q^i R_i(y) \quad \text{and} \quad k_2 = q^i R_i(x).$$

It follows from (1.6), (2.2) and (2.3) that

$$(2.11) \quad \{y\}_i = T^{k_1}(0), \quad \{x\}_i = T^{k_2}(0)$$

and

$$(2.12) \quad x = T^{k_2}(\{q^i x\}/q^i).$$

Then, in view of (1.6),

$$(2.13) \qquad \{q^i T^{k_0}(x)\} = \begin{cases} \{q^i x\}, & \text{if } k_2 + k_0 < q^i, \\ T\{q^i x\}, & \text{otherwise.} \end{cases}$$

By (2.6), (2.11) and (2.12), we obtain

$$\{T^{k_2+k_0}(\{q^i x\}/q^i)\}_i = T^{k_1}(0).$$

Next, from (1.6) we conclude that

$$k_1 \equiv k_2 + k_0 \pmod{q^i}$$

and

$$k_2 + k_0 < q^i \Leftrightarrow k_2 \leq k_1, \quad \text{for } k_1, k_2, k_0 \in [0, q^i).$$

Using (2.10), we obtain from (2.13)

$$(2.14) \qquad \{q^i T^{k_0}(x)\} = \begin{cases} \{q^i x\}, & \text{if } R_i(x) \leq R_i(y), \\ T(\{q^i x\}), & \text{otherwise.} \end{cases}$$

Now in view of (2.9), the assertion of the lemma follows.   ∎

LEMMA 2.3: *Let $i \geq 1$, $a \geq 0$ be integers and $v, y \in [0,1)$. Then*

$$
(2.15) \quad \sum_{k=aq^i}^{(a+1)q^i - 1} (1_{[0,y)}(T^k(v)) - y) = -\{q^i y\} + I(R_i(x) \leq R_i(y))
$$
$$
\times I(T^a(q^i x) < \{q^i y\}) + I(R_i(x) > R_i(y))
$$
$$
\times I(T^{a+1}(q^i x) < \{q^i y\}).
$$

Proof: We denote the left side of (2.15) by σ. Bearing in mind that $T^{aq^i + k}(v) = T^b(T^{aq^i}(v))$, we apply Lemma 2.2 with $x = T^{aq^i}(v)$:

$$
\sigma = -\{q^i y\} + I(R_i(T^{aq^i}(v)) \leq R_i(y)) I(\{q^i T^{aq^i}(v)\} < \{q^i y\})
$$
$$
+ I(R_i(T^{aq^i}(v)) > R_i(y)) I(T\{q^i T^{aq^i}(v)\} < \{q^i y\}).
$$

Now by Lemma 2.1, (1.6), and (2.2), we obtain

$$R_i(T^{aq^i}(v)) = R_i(v),$$
$$\{q^i T^{aq^i}(v))\} = T^a(\{q^i v\}),$$

and the assertion of the lemma follows.   ∎

Consider the q-expansion of the integer number $M \in [q^m, q^{m+1})$,

$$(2.16) \qquad M = \sum_{i=0}^{m} a_i q^i, \quad \text{with } a_i \in \{0, \ldots, q-1\}, \; i = 0, \ldots, m-1,$$

Let

$$(2.17) \qquad M_i = [M/q^{i+1}]q = \sum_{k=i+1}^{m} a_k q^{k-i}, \quad i = 0, \ldots, m.$$

It is easy to see that

$$(2.18) \qquad [0, M) = \bigcup_{i=0}^{m} \bigcup_{b_i=0}^{a_i-1} [q^i(b_i + M_i), q^i(b_i + 1 + M_i)).$$

Define

$$
\begin{aligned}
g_i(L, x, y) = &- \{q^i y\} + I(R_i(x) \le R_i(y)) I(T^L(q^i x) < \{q^i y\}) \\
(2.19) \qquad &+ I(R_i(x) > R_i(y)) I(T^{L+1}(q^i x) < \{q^i y\}) \quad \text{for } i = 1, 2, \ldots
\end{aligned}
$$

and

$$(2.20) \qquad g_0(L, x, y) = 1_{[0,y)}(T^L(x)) = y.$$

We note that $g_i(L, x, y)$ is the function periodic with period 1 over arguments x and y $(i = 0, 1, 2, \ldots)$.

From Lemma 2.3, we have

COROLLARY 2.1: *Let $x, y \in [0, 1)$, and M be an integer. Then in notation (2.16) and (2.17)*

$$(2.21) \qquad \sum_{k=0}^{M-1} (1_{[0,y)}(T^k(x)) - y) = \sum_{i=0}^{m} \sum_{b_i=0}^{a_i-1} g_i(b_i + M_i, x, y).$$

LEMMA 2.4: *Let $x, y, u \in [0, 1)$. Then*

$$(2.22) \qquad 1_{[u, u+y)}(x) - u = 1_{[0, \{u+y\})}(x) - \{u+y\} - 1_{[0,u)}(x) + u.$$

Proof: Define

$$x \in \Delta \pmod 1 \quad \text{if } \exists\, \ell \in \mathbb{Z} : x + \ell \in \Delta.$$

According to (1.1) and (1.5)

$$
\begin{aligned}
(2.23) \qquad & 1_{[u, u+y)}(x) = 1_{[0,y)}(\{x - u\}) = 1 \Leftrightarrow \{x - u\} \in [0, y) \\
& \qquad \Leftrightarrow x - u \in [0, y) \pmod 1 \Leftrightarrow x \in [u, u+y) \pmod 1.
\end{aligned}
$$

Consider two cases:

CASE 1: $u + y < 1$.

We see that $\{u + y\} = u + y$, and

$$1_{[u,u+y)}(x) = 1 \Leftrightarrow x \in [u, u + y).$$

Hence

$$1_{[u,u+y)}(x) = 1_{[0,u+y)}(x) = 1_{[0,u)}(x)$$

and (2.22) follows.

CASE 2: $u + y \geq 1$.

By (2.23)

$$1_{[u,u+y)}(x) = 1_{[u,1)}(x) + 1_{[0,\{u+y\})}(x) = 1_{[0,\{u+y\})}(x) + 1 - 1_{[0,u)}(x).$$

Bearing in mind that $\{u + y\} = u + y - 1$ for this case, we now obtain (2.22). The lemma is proved. ∎

Let

$$(2.24) \qquad f(M, x, y, u) = \sum_{k=0}^{M-1} (1_{[0,\{y\})}(\{T^k(x) - u\}) - \{y\}).$$

LEMMA 2.5: *Let* $x, y, u \in [0, 1)$, $M \geq 1$ *be an integer. Then in notations (2.16) and (2.17)*

$$(2.25) \qquad f(M, x, y - u, u) = \sum_{i=0}^{m} \xi_i(M, x, y, u),$$

where

$$(2.26) \qquad \xi_i(M, x, y, u) = \sum_{b_i=0}^{a_i-1} (g_i(b_i + M_i, x, y) - g_i(b_i + M_i, x, u)).$$

Proof: In view of Lemma 2.4, and (1.5) and that

$$\{u + \{y - u\}\} = \{u + y - u - [y - u]\} = \{y - [y - u]\} = y,$$

we obtain

$$f(M, x, y - u, u) = \sum_{k=0}^{M-1} (1_{[0,\{y-u\})}(\{T^k(x) - u\}) - \{y - u\})$$

$$= \sum_{k=0}^{M-1} (1_{[u,u+\{y-u\})}(T^k(x)) - \{y - u\})$$

$$= \sum_{k=0}^{M-1} (1_{[0,\{u+\{y-u\}\}]}(T^k(x)) - \{u + \{y - u\}\}$$
$$- 1_{[0,u)}(T^k(x)) + u)$$
$$= \sum_{k=0}^{M-1} (1_{[0,y)}(T^k(x)) - y) - \sum_{k=0}^{M-1} (1_{[0,u)}T^k(x) - u).$$

Now from (2.21) and (2.26), we obtain the desired result.   ∎

Thus $f(N, x, y - u, u)$ is the sum of dependent random variables $(\xi_i)_{i=0}^n$ with $n = [\log_q N]$. To prove Theorems 1.1–1.3 we will use Bernstein's method [Be], approximating f by the sum of independent random variables $\xi_{i,k}^{(\mu)}$ $(\mu = 1, 2, 3)$: We define

$$(2.27) \qquad \xi_{i,k}^{(\mu)}(M, x, y, u) = \sum_{b_i=0}^{a_i-1} (g_{i,k}(b_i + M_i, x, y) - g_{i,k}^{(\mu)}(b_i + M_i, x, u)),$$

with notations (2.16)–(2.18), where

$$g_{i,k}(L, x, v) = -\{q^i v\}_k + I(\{R_i(x)\}_k \le \{R_i(v)\}_k)I(\{T^L(q^i x)\}_k < \{q^i v\}_k)$$
$$(2.28) \qquad + I(\{R_i(x)\}_k > \{R_i(v)\}_k)I(T^{L+1}\{q^i x\}_k),$$
$$(2.29) \qquad g_{i,k}^{(\mu)}(L, x, v) = g_{i,k}(L, x, v), \quad \text{for } \mu = 2, 3,$$

and
$$(2.30)$$
$$g_{i,k}^{(1)}(L, x, v) = -\{q^i v\} + I(R_i(x) \le R_i(v))I(\{T^L(q^i x)\}_k < \{q^i v\}_k)$$
$$+ I(R_i(x) > R_i(v))I(\{T^{L+1}(q^i x)\}_k < \{q^i v\}_k), \quad \text{for } i, k \ge 1.$$

LEMMA 2.6: *Let $i \ge k \ge 1$, L be integers;*

$$L = \sum_{j=0}^{k} c_j q^j, \quad c_j \in \{0, \dots, q-1\}, \quad x, v \in [0, 1); \; x = .x_1 x_2 \dots,$$

and $v = .v_1 v_2 \dots$. Then $g_{i,k}(L, x, v)$ depend only on

$$(x_{i-k+1}, \dots, x_{i+k}), \; (v_{i-k+1}, \dots, v_{i+k}), \quad \text{and} \quad (c_0, \dots, c_{k-1});$$

and $g_{i,k}^{(1)}(L, x, v)$ depend only on x, v, i and $(c_0, \dots, c_{k-1})$.

Proof: By (2.2) and (2.3) we see that

$$(2.31) \qquad\qquad \{q^i v\}_k = .v_{i+1} \dots v_{i+k}$$

and

$$(2.32) \qquad\qquad \{R_i(v)\}_k = .v_1 v_{i-1} \ldots v_{i-k+1}.$$

From (1.6) and Lemma 2.1, we obtain

$$\{T^L(q^i x)\}_k = \{T^L(.x_{i+1} \ldots x_{i+k})\}_k = \{T^{L_1}(.x_{i+1} \ldots x_{i+k})\}_k$$

where $L_1 = \sum_{j=0}^{k-1} c_j q^j$.

Thus, by (2.28) we have that $g_{i,k}(L, x, v)$ depend only on $(x_{i-k+1}, \ldots, x_{i+k})$, $(v_{i-k+1}, \ldots, v_{i+k})$ and $(c_0, \ldots, c_{k-1})$. Similarly, from (2.30) we have that $g_{i,k}^{(1)}(L, x, v)$ depend only on x, i, v, i and $(c_0, \ldots, c_{k-1})$. In view of (2.27), the lemma is proved. ∎

LEMMA 2.7: *Let $L \geq 0$, $i \geq k_1 \geq k \geq 1$ be integers, $(x, y) \in [0, 1)$. Then*

$$(2.33) \qquad \operatorname{mes} \left\{ y \in [0, 1) \big| g_{i,k}(L, x, y) - g_i(L, x, y) \notin \left[0, \frac{1}{q^k} \right) \right\} \leq \frac{2}{q^k}$$

and

$$(2.34) \qquad \operatorname{mes} \left\{ y \in [0, 1) \big| g_{i,k}(L, x, y) - g_{i,k_1}(L, x, y) \notin \left[0, \frac{1}{q^k} \right) \right\} \leq \frac{2}{q^k}.$$

Proof: We will prove only the inequality (2.33). The proof of (2.34) is similar to that of (2.33).

Let $y = .y_1 y_2 \ldots$ be the q-expansion of y,

$$y^{(1)} = .y_1 \ldots y_{i-k} \in \Omega_1 = \{.v_1 \ldots v_{i-k} | v_j \in \{0, \ldots, q-1\}, j = 1, \ldots, i-k\},$$
$$y^{(2)} = .y_i y_{i-1} \ldots y_{i-k-1} \in \Omega_2 = \{.v_1 \ldots v_k | v_j \in \{0, \ldots, q-1\}, j = 1, \ldots, k\},$$
$$y^{(3)} = .y_{i+1} \ldots y_{i+k} \in \Omega_2,$$
$$y^{(4)} = .y_{i+k+1} y_{i+k+2} \ldots \in \Omega_3 = \{.v_1 v_2 \ldots | v_j \in \{0, \ldots, q-1\}, j = 1, 2 \ldots\}.$$

Using Fubini's theorem, we obtain that the left hand side of (2.33) is equal to

$$\int_{\Omega_1} \int_{\Omega_3} \int_{\Omega_2} \int_{\Omega_2} I\left(g_{i,k}(L, x, y) - g_i(L, x, y) \notin \left[0, \frac{1}{q^k} \right) \right) dw,$$

with $dw = dy^{(2)} dy^{(3)} dy^{(4)} dy^{(1)}$.

Hence, to prove (2.33) it is sufficient to verify that

$$(2.35) \qquad \int_{\Omega_2} \int_{\Omega_2} I\left(g_{i,k}(L, x, y) - g_i(L, x, y) \in \left[0, \frac{1}{q^k} \right) \right) dy^{(2)} dy^{(3)} \leq \frac{2}{q^k}$$

for all $y^{(1)} \in \Omega_1$ and $y^{(4)} \in \Omega_3$.

It is easy to see that $I(\{R_i(x)\}_k = \{R_i(y)\}_k) = 1$ holds only for one $y_0^{(2)} \in \Omega_2$ (see (2.2)). Bearing in mind that

$$\mathrm{mes}\{y^{(2)} \in \Omega_2 \mid y^{(2)} = y_0^{(2)}\} = 1/q^k,$$

we get that to obtain (2.35), it is sufficient to prove that

$$(2.36) \qquad \int_{\Omega_2} I\Big(g_{i,k}(L,x,y) - g_i(L,x,y) \notin \Big[0, \frac{1}{q^k}\Big)\Big)\,dy^{(3)} \le \frac{1}{q^k}$$

for all $y^{(1)} \in \Omega_1$, $y^{(4)} \in \Omega_3$ and $y^{(2)} \in \Omega_2 \smallsetminus y_0^{(2)}$. If $(\{R_i(x)\}_k < \{R_i(y)\}_k) = 1$, then by (2.2) and (2.3), $I(R_i(x) < R_i(y)) = 1$, and by (2.19) and (2.28),

$$(2.37) \quad \begin{aligned} g_{i,k}(L,x,y) - g_i(L,x,y) - \{q^i y\} + \{q^i y\}_k = & I(\{T^L(q^i x)\}_k < \{q^i y\}_k) \\ & - I(T^L(q^i x) < \{q^i y\}). \end{aligned}$$

We see that the right side of (2.37) can be not equal to zero only for the case

$$I(\{T^L(q^i x)\}_k = \{q^i y\}_k) = 1.$$

This equality holds only for one $y^{(3)} \in \Omega_2$. Bearing in mind that $\{R_i(y)\}_k$ does not depend on $y^{(3)}$, we obtain (2.36) from (2.37) and (2.3). Now let

$$I(\{R_i(x)\}_k > \{R_i(y)\}_k) = 1.$$

We have that

$$I(R_i(x) > R_i(y)) = 1,$$

and by (2.19) and (2.28), the left side of (2.37) is equal to

$$(2.38) \qquad I(\{T^{L+1}(q^i x)\}_k < \{q^i y\}_k) - I(T^{L+1}(q^i x) < \{q^i y\}).$$

This difference can be not equal to zero only for one $y^{(3)} = y_0^{(3)} \in \Omega_2$. Now from (2.37), we get

$$g_{i,k}(L,x,y) - g_i(L,x,y) = \{q^i y\} - \{q^i y\}_k \in [0, 1/q^k) \quad \text{for } y^{(3)} \ne y_0^{(3)}.$$

Thus, we obtain (2.36) and the assertion of the lemma. ∎

LEMMA 2.8: *Let $a \ge 0$, $i \ge k_1 \ge k \ge 1$ be integers, $x, y \in [0,1)$. Then*

$$(2.39) \qquad \frac{1}{q^k} \sum_{L=0}^{q^k-1} I(g_{i,k}^{(1)}(L + aq^k, x, y) \ne g_i(L + aq^k, x, y)) \le \frac{1}{q^k}$$

and

$$(2.40) \qquad \frac{1}{q^k} \sum_{L=0}^{q^k-1} I(g_{i,k}^{(1)}(L + aq^k, x, y) \neq g_{i,k_1}(L + aq^k, x, y)) \leq \frac{1}{q^k}.$$

Proof: We consider (2.39). Let

$$I(R_i(x) \leq R_i(y)) = 1.$$

Then by (2.19) and (2.30), we have that

$$(2.41) \qquad \begin{aligned} g_{i,k}^{(1)}(L + aq^k, x, y) - g_i(L + aq^k, x, y) = &I(\{T^{L+aq^k}(q^i x)\}_k < \{q^i y\}_k) \\ &- I(T^{L+aq^k}(q^i x) < \{q^i y\}). \end{aligned}$$

It is easy to see that if, for $s \geq 0$,

$$I(\{T^s(q^i x)\}_k < \{q^i y\}_k) = 1,$$

then

$$I(T^s(q^i x) < \{q^i y\}) = 1,$$

and if

$$I(\{T^s(q^i x)\}_k > \{q^i(y)\}_k) = 1,$$

then

$$I(T^s(q^i x) > \{q^i x\}) = 1.$$

Hence the right hand side of (2.41) can be not equal to zero only for the case

$$(2.42) \qquad I(\{T^{L+aq^k}(q^i x)\}_k = \{q^i y\}_k) = 1.$$

By Lemma 2.1 and (1.6) this equality holds only for one integer $L \in [0, q^k)$, and (2.39) follows.

By (2.19) and (2.30), we obtain (2.39) analogously for the case

$$I(R_i(x) > R_i(y)) = 1.$$

The proof of (2.40) is similar to that of (2.39).   ∎

Let

$$(2.43) \qquad\qquad N = \sum_{i=0}^{n} c_i q^i$$

be the q-expansion of N, $c_n \neq 0$, $c_i \in \{0, \ldots, q-1\}$, $i = 0, \ldots, n$, and

$$(2.44) \qquad N_L = q[N/q^{i+1}], \quad i = 0, \ldots, n.$$

In what follows, we fix $N \in [q^n, q^{n+1})$ and the sequence of integers $d_i \in [0, c_i)$, $i = 0, 1, \ldots, n$. We define for $\mu \in \{1, 2, 3\}$

$$
\begin{aligned}
E_j^{(\mu)} h = E_j^{(\mu)} h(M, x, y, u) &= E_{j,(N_j+d_j)q^j}^{(\mu)} h(M, x, y, u) \\
(2.45) \qquad\qquad &= \frac{1}{q^j} \sum_{M=0}^{q^j-1} \int_{G_\mu} h(M + (N_j + d_j)q^j, x, y, u) dw_\mu,
\end{aligned}
$$

where

$$(2.46) \qquad G_1 = [0,1), G_2 = G_3 = [0,1)^2; \quad dw_1 = dy, dw_2 = dydu,$$

and $dw_3 = dydx$;

$$
\begin{aligned}
\mathrm{Cov}_j^{(\mu)}(h_1, h_2) &= \mathrm{Cov}_j^{(\mu)}(h_1(M, x, y, u), h_2(M, x, y, u)) \\
&= \mathrm{Cov}_{j,(N_j+d_j)q^j}^{(\mu)}(h_1(M, x, y, u), h_2(M, x, y, u)) \\
(2.47) \qquad &= E_{j,(N_j+d_j)q^j}^{(\mu)}(h_1(M, x, y, u) - E_{j,(N_j+d_j)q^j}^{(\mu)} h_1(M, x, y, u)) \\
&\quad \times (h_2(M, x, y, u) - E_{j,(N_j+d_j)q^j}^{(\mu)} h_2(M, x, y, u)))
\end{aligned}
$$

and

$$
\begin{aligned}
D_j^{(\mu)} h &= D_{j,(N_j+d_j)q^j}^{(\mu)} h(M, x, y, u) \\
(2.48) \qquad &= \mathrm{Cov}_{j,(N_j+d_j)q^j}^{(\mu)}(h(M, x, y, u), h(M, x, y, u)).
\end{aligned}
$$

LEMMA 2.9: *Let* $i \geq k_1 \geq k \geq 1$, $j \geq i + k_1$, $\mu \in \{1, 2, 3\}$. *Then*

$$(2.49) \qquad E_j^{(\mu)}|\xi_{i,k}^{(\mu)} - \xi_i| \leq 15/q^{k-1}$$

and

$$(2.50) \qquad E_j^{(\mu)}|\xi_{i,k}^{(\mu)} - \xi_{i,k_1}^{(\mu)}| \leq 15/q^{k-1}.$$

Proof: We will prove the inequality (2.49). The proof of (2.50) is similar to that of (2.49). Using (2.17), (2.26), (2.27), (2.29) and (2.30), we obtain

$$
\begin{aligned}
|\xi_{i,k}^{(\mu)}&((M + (N_j + d_j)q^j, x, y, u) - \xi_i(M + (N_j + d_j)q^j, x, y, u)| \\
(2.51) \qquad &\leq \sum_{b=0}^{q-1}(|g_{i,k}(b + M_i', x, y) - g_i(b + M_i', x, y)| \\
&\qquad + |g_{i,k}(b + M_i', x, u) - g_i(b + M_i', x, u)| \\
&\qquad + |g_{i,k}^{(1)}(b + M_i', x, u) - g_i(b + M_i', x, u)|)
\end{aligned}
$$

where

(2.52) $$M_i' = q[M/q^{i+1}] + (N_j + d_j)q^{j-i}.$$

In view of (2.19), (2.28) and (2.30), we have that

(2.53) $$|g_{i,k}(L, x, y) - g_i(L, x, y)| \leq 2$$

and

(2.54) $$|g_{i,k}^{(1)}(L, x, u) - g_i(L, x, u)| \in \{0, 1\}.$$

From Lemma 2.7 and (2.53), we conclude that

(2.55)
$$\int_0^1 |g_{i,k}(L, x, y) - g_i(L, x, y)| dy$$
$$\leq \frac{1}{q^k} + L \int_0^1 I\Big(g_{i,k}(L, x, y) - g_i(L, x, y) \notin \Big[0, \frac{1}{q^k}\Big)\Big) dy$$
$$\leq 5/q^k.$$

Applying Lemma 2.8 and (2.54), we see that

(2.56)
$$\frac{1}{q^k} \sum_{L=0}^{q^k-1} |g_{i,k}^{(1)}(L + aq^k, x, u) - g_i(L + aq^k, x, u)|$$
$$\leq \frac{1}{q^k} \sum_{L=0}^{q^k-1} I(g_{i,k}^{(1)}(L + aq^k, x, u) \neq g_i(L + aq^k, x, u))$$
$$\leq \frac{1}{q^k} \quad \text{for } a \geq 0.$$

Since $j \geq i + k$, from (2.45) and (2.52)–(2.56) we get that

$$E_{j,(N_j+d_j)q^j}^{(\mu)} |g_i^{(\mu)}(M, x, y) - g_i(M, x, y)| \leq 5/q^k \quad \text{for } \mu \in [1, 3].$$

Now, by (2.51) we obtain the assertion of the lemma.   ∎

LEMMA 2.10: *Let* $\nu \geq 0$, $i \geq 1$, $\mu \in [1, 3]$, $k \geq \nu_1 = \min([\nu/2], [2\log_q n + 1])$, $j \geq i + \nu + \nu_1$. *Then*

(2.57) $$|\operatorname{Cov}_j^{(\mu)}(\xi_i, \xi_{i+\nu})| \leq 60q^{2-\nu_1}$$

and

(2.58) $$|\operatorname{Cov}_j^{(\mu)}(\xi_{i,k}^{(\mu)}, \xi_{i+\nu,k}^{(\mu)})| \leq 60q^{2-\nu_1}.$$

Proof: We will prove (2.57). The proof of (2.58) is similar to that of (2.57). If $0 \leq \nu \leq 3$, then (2.57) follows from (2.19) and (2.26). Now let $\nu \geq 4$,

$$(2.59) \qquad \omega_1^{(\mu)} = \xi_i - \xi_{i,\nu_1}^{(\mu)} \quad \text{and} \quad \omega_2^{(\mu)} = \xi_{i+\nu} - \xi_{i+\nu,\nu_1}^{(\mu)}.$$

We have that

$$
\begin{aligned}
(2.60) \qquad \operatorname{Cov}_j^{(\mu)}(\xi_i, \xi_{i+\nu}) &= \operatorname{Cov}_j^{(\mu)}(\xi_{i,\nu_1}^{(\mu)} + \omega_1^{(\mu)}, \xi_{i+\nu,\nu_1}^{(\mu)} + \omega_2^{(\mu)}) \\
&= \operatorname{Cov}_j^{(\mu)}(\xi_{i,\nu_1}^{(\mu)}, \xi_{i+\nu,\nu_1}^{(\mu)}) + \operatorname{Cov}_j^{(\mu)}(\omega_1^{(\mu)}, \xi_{i+\nu}) \\
&\quad + \operatorname{Cov}_j^{(\mu)}(\xi_{i,\nu_1}^{(\mu)}, \omega_2^{(\mu)}).
\end{aligned}
$$

Let $\mu \in \{2, 3\}$. From (2.27), (2.29) and Lemma 2.6, we obtain that $\xi_{i,\nu_1}^{(\mu)}(M, x, y, u)$ depend only on $((x_r, y_r, u_r))_{r=i-\nu_1+1}^{i+\nu_1}$ and $(a_i, \ldots, a_{i+\nu_1-1})$; $\xi_{i+\nu,\nu_1}^{(\mu)}(M, x, y, u)$ depend only on

$$((x_r, y_r, u_r))_{r=i+\nu-\nu_1+1}^{i+\nu+\nu_1} \quad \text{and} \quad (a_{i+\nu}, \ldots, a_{i+\nu+\nu_1-1}).$$

We see that $i + \nu_1 < i + \nu - \nu_1 + 1$. Hence the random variables $\xi_{i,\nu_1}^{(\mu)}$ and $\xi_{i+\nu,\nu_1}^{(\mu)}$ are independent. Bearing in mind that $j \geq i + \nu_1$, we get from (2.47) that

$$(2.61) \qquad \operatorname{Cov}_j^{(\mu)}(\xi_{i,\nu_1}^{(\mu)}, \xi_{i+\nu,\nu_1}^{(\mu)}) = 0.$$

Now we will prove that (2.61) is also valid for the case $\mu = 1$. From (2.27), (2.30) and Lemma 2.6, it follows that for fixed $x, u \in [0, 1)$, the random variable $\xi_{i,\nu_1}^{(1)}$ depends only on $(y_{i\nu_1+1}, \ldots, y_{i+\nu_1})$ and $(a_i, \ldots, a_{i+\nu_1-1})$; the random variable $\xi_{i+\nu,\nu_1}^{(1)}$ depends only on $(y_{i+\nu-\nu_1+1}, \ldots, y_{i+\nu+\nu_1})$ and $(a_{i+\nu}, \ldots, a_{i+\nu+\nu_1-1})$. Hence the random variables $\xi_{i,\nu_1}^{(1)}$ and $\xi_{i+\nu,\nu_1}^{(1)}$ are independent. Taking into account that $j \geq i + \nu_1$, we obtain (2.61). In view of (2.9), (2.26) and (2.27)–(2.30), we have that $|\xi_{i+\nu}^{(\mu)}| \leq 2q$, $|\xi_{i,\nu_1}^{(\mu)}| \leq 2q$.

Then, by Lemma 2.9, (2.47) and (2.59),

$$
\begin{aligned}
|\operatorname{Cov}_j^{(\mu)}(\omega_1^{(\mu)}, \xi_{i+\nu})| &= |E_j^{(\mu)}(\xi_{i+\nu} - E_j^{(\mu)}(\xi_{i+\nu}))\omega_1^{(\mu)}| \leq 2q E_j^{(\mu)}|\omega_1^{(\mu)}| \\
&= 2q E_j^{(\mu)}|\xi_i - \xi_{i,\nu_1}^{(\mu)}| \leq 30/q^{\nu_1-2}
\end{aligned}
$$

and

$$|\operatorname{Cov}_j^{(\mu)}(\xi_{i,\nu_1}^{(\mu)}, \omega_2^{(\mu)})| \leq 2q E_j^{(\mu)}|\omega_2^{(\mu)}| = 2q E_j^{(\mu)}|\xi_{i+\nu} - \xi_{i+\nu,\nu_1}^{(\mu)}| \leq 30/q^{\nu_1-2}.$$

Now from (2.60) and (2.61), we obtain the assertion of the lemma.   ∎

Let

$$(2.62) \qquad k_0 = [2 \log_q n + 1].$$

LEMMA 2.11: *Let $\mu \in [1,3]$, $m_1 \geq 0$, $m_2 \geq 1$, $k \geq k_0$, and $n \geq j \geq m_1 + m_2 + k_0$. Then*

$$(2.63) \qquad D_j^{(\mu)}\left(\sum_{i=m_1+1}^{m_1+m_2} \xi_i \right) \leq 2^{10} q^2 m_2$$

and

$$(2.64) \qquad D_j^{(\mu)}\left(\sum_{i=m_1+1}^{m_1+m_2} \xi_{i,k}^{(\mu)} \right) \leq 2^{10} q^2 m_2.$$

Proof: Consider (2.63). Using Lemma 2.10, we obtain that the left hand side of (2.63) is less than

$$2\sum_{i=m_1+1}^{m_1+m_2} \sum_{\nu=0}^{m_1+m_2-i} |\operatorname{Cov}_j^{(\mu)}(\xi_i, \xi_{i+\nu})| \leq 2 \sum_{i=m_1+1}^{m_1+m_2} \sum_{\nu=0}^{m_1+m_2-i} 60 q^2 \max\left(q^{-[\nu/2]}, \frac{1}{n^2} \right)$$

$$\leq 120 q^2 m_2 \left(\frac{2}{1 - q^{-1/2}} + \frac{1}{n} \right) \leq 2^{10} q^2 m_2.$$

Similarly, (2.64) follows from (2.58), and the lemma is proved. ∎

Let

$$(2.65) \qquad d_1 = [n^{2/3}] + 1, \quad d_1 = [n/d_1], \quad j_0 = n - [2\log_q n] - 1 = n - k_0;$$

$$(2.66) \qquad \zeta_i^{(\mu)} = \sum_{\nu \in \Delta_i} \xi_{\nu,k_0}^{(\mu)}(M,x,y,u), \quad 0 \leq i \leq d_2,$$

where

$$(2.67) \qquad \Delta_i = (i d_1 + k_0, (i+1)d_1 - 2k_0), \quad 0 \leq i \leq d_2 - 1$$

and

$$(2.68) \qquad \Delta_{d_2} = \begin{cases} (d_1 d_2 + k_0, j_0 - 2k_0], & \text{if } j_0 \geq d_1 d_2 + 3k_0 + 1, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$\hat{\omega}_j^{(\mu)} = \omega^{(\mu)} - E_j^{(\mu)}\omega, \quad \mu \in [1,3], \quad j = 1, 2, \ldots.$$

LEMMA 2.12: *Let $0 \leq i \leq d_2$, $j > j_0$, and $\mu \in [1,3]$. Then*

$$E_j^{(\mu)}(\hat{\zeta}_i^{(\mu)})^4 \leq 2^9 q^4 d_1^2 (2k_0 + 1)^2.$$

Proof: Let

$$a_{\nu_1\nu_2\nu_3\nu_4}^{(\mu)} = E_j^{(\mu)}(\hat{\xi}_{\nu_1,k_0}^{(\mu)} \hat{\xi}_{\nu_2,k_0}^{(\mu)} \hat{\xi}_{\nu_3,k_0}^{(\mu)} \hat{\xi}_{\nu_4,k_0}^{(\mu)}).$$

It is easy to see that

$$E_j^{(\mu)}(\hat{\zeta}_i^{(\mu)})^4 \le 4! \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4 \in \Delta_i \\ \nu_1 \le \nu_2 \le \nu_3 \le \nu_4}} |a_{\nu_1 \nu_2 \nu_3 \nu_4}^{(\mu)}|.$$

Using Lemma 2.6, we obtain that if $\nu_2 - \nu_1 > 2k_0$ $(\nu_4 - \nu_3 > 2k_0)$ then $\xi_{\nu_1,k_0}^{(\mu)}$ is not dependent on $(\xi_{\nu_2,k_0}^{(\mu)}, \xi_{\nu_3,k_0}^{(\mu)}, \xi_{\nu_4,k_0}^{(\mu)})$ (correspondingly, $\xi_{\nu_4,k_0}^{(\mu)}$ is not dependent on $(\xi_{\nu_1,k_0}^{(\mu)}, \xi_{\nu_2,k_0}^{(\mu)}, \xi_{\nu_3,k_0}^{(\mu)})$). Hence $a_{\nu_1 \nu_2 \nu_3 \nu_4} = 0$ if $\nu_2 - \nu_1 > 2k_0$ or $\nu_4 - \nu_3 > 2k_0$.

Bearing in mind that $|\hat{\xi}_{\nu,k_0}^{(\mu)}| \le 2q$ (see (2.27)), we get that

$$E_j^{(\mu)}(\hat{\zeta}_i^{(\mu)})^4 \le 4! \sum_{\nu_1,\nu_4 \in \Delta_i} \sum_{\substack{\nu_2,\nu_3 \in \Delta_i \\ \nu_2 - \nu_1, \nu_4 - \nu_3 \in [0,2k_0]}} (2q)^4 \le 4!(2q)^4 d_1^2 (2k_0 + 1)^2.$$

This is the desired result. ∎

LEMMA 2.13: *Let* $j \in [j_0, n]$, $j_0 = n - [2\log_q n] - 1$, *and* $\mu \in [1,3]$. *Then*

$$(2.69) \qquad D_j^{(\mu)}(f(M, x, y - u, u)) \ge 2^{-10} j \log_2 q, \quad \text{for } n \ge n_0(q).$$

Proof: Let $M \in [0, q^j)$, and

$$M + (N_j + d_j)q^j = \sum_{i=0}^n a_i q^i, \quad \text{with } a_i \in [0, q-1),$$

$$M_i = [M/q^{i+1}]q, \quad M_i' = [(M + (N_j + d_j)q^j)/q^{i+1}]q, \quad i = 0, 1, 2, \ldots.$$

By (2.25) and (2.26), we see that
(2.70)

$$f(M + (N_j + d_j)q^j, x, y - u, u) = \sum_{i=0}^n \xi_i(M + (N_j + d_j)q^j, x, y, u)$$

$$= \sum_{i=0}^n \sum_{b_i=0}^{a_i-1} (g_i(b_i + M_i', x, y) - g_i(b_i + M_i', x, u)).$$

From (2.19) we obtain for $i \in [1, j)$
(2.71)

$$g_i(b_i + M_i', x, y) = - \{q^i y\} + I(R_i(x) \le R_i(y))I(T^{b_i + M_i + N_j' q^{j-i}}(q^i x)) < \{q^i y\})$$

$$+ I(R_i(x) > R_i(y))I(T^{1 + b_i + M_i + N_j' q^{j-i}}(q^i x) < \{q^i y\}).$$

It is easy to see for $x, \varepsilon \in [0, 1)$, and $i < j$ that
(2.72)

$$\left\{ \frac{[q^{j-i}\{q^i x\}] + \varepsilon}{q^{j-i}} \right\} = \left\{ \frac{[q^j x - q^{j-i}[q^i x]] + \varepsilon}{q^{j-i}} \right\} = \left\{ \frac{[q^j x] - q^{j-i}[q^i x] + \varepsilon}{q^{j-i}} \right\}$$

$$= \left\{ q^i \left\{ \frac{[q^j x] + \varepsilon}{q^j} \right\} \right\}.$$

In view of Lemma 2.1, we have that

$$(2.73) \qquad T^{L+N_j' q^{j-i}}(q^i x) = T^L\left(\frac{[q^{j-i}\{q^i x\}] + T^{N_j'}(q^j x)}{q^{j-i}}\right) = T^L(q^i x_{N,j}),$$
$$\text{with } L = b_i + M_i, b_i + M_i + 1,$$

where, according to (2.72),

$$(2.74) \qquad x_{N,j} = \left\{\frac{[q^j x] + T^{N_j'}(q^j x)}{q^j}\right\},$$

and we find that

$$[q^i x_{N,j}] = [q^i x], \quad \text{for } 1 \le i < j.$$

Thus, by (2.2),

$$(2.75) \qquad R_i(x_{N,j}) = R_i(x), \quad \text{for } 1 \le i < j.$$

Now, from (2.71), (2.73), (2.75) and (2.26), we conclude that

$$g_i(b_i + M_i', x, y) = g_i(b_i + M_i, x_{N,j}, y),$$

and

$$(2.76) \qquad \xi_i(b_i + M_i', x, y, u) = \xi_i(b_i + M_i, x_{N,j}, y, u) \quad \text{for } 1 \le i < j.$$

Bearing in mind that $|\xi_i(M, x, y, u)| \le 2q$, we obtain

$$(2.77) \qquad \begin{aligned} &|f(M + (N_j + d_j)q^j, x, y - u, u) - f(M, x_{N,j}, y - u, u)| \\ &= \left|\sum_{i \in \{0\} \cup [j,n]} \xi_i(M + (N_j + d_j)q^j, x, y, u) - \xi_i(M, x_{N,j}, y, u)\right| \\ &\le 2q(n - j + 2). \end{aligned}$$

By (2.77) and the inequality $2a^2 + 2b^2 \ge (a - b)^2$, we have that

$$(2.78) \qquad \begin{aligned} D_{j,(N_j + d_j)q^j}^{(\mu)}(f(M, x, y - u, u)) \ge &\frac{1}{2} D_{j,0}^{(\mu)}(f(M, x_{N,j}, y - u, u)) \\ &- 4q^2(n - j + 2)^2. \end{aligned}$$

If follows from (2.24) that the function $f(M, x, y, u)$ is periodic over y with period

1. Hence, by (2.45)–(2.48), we obtain

$$D_{j,0}^{(\mu)}(f(M, x_{N,j}, y - u, u))$$

$$= D_{j,0}^{(\mu)}(f(M, x_{N,j}, y, u))$$

$$= \frac{1}{q^j} \sum_{M=0}^{q^j-1} \int_{G_\mu} \left(\sum_{k=0}^{M-1} (1_{[0,y)}(\{T^k(x_{N,j}) - u\}) - y) - E_{j,0}^{(\mu)} f(M, x_{N,j}, y, u) \right)^2 dw_\mu$$

$$\geq \inf_{x,u\in[0,1)} \inf_{\rho\in\mathbb{R}} \frac{1}{q^j} \sum_{M=0}^{q-1} \int_0^1 \left(\sum_{k=0}^{M-1} (1_{[0,y)}(\{T^k(x) - u\}) - y) - \rho \right)^2 dy.$$

Using (1.2) and Fubini's theorem, we obtain

$$D_{j,0}^{(\mu)}(f(M, x_{N,j}, y - u, u))$$

$$\geq \inf_{x,y\in[0,1)} \inf_{\rho\in\mathbb{R}} \int_0^1 \int_0^1 \left(\sum_{k=0}^{\lfloor zq^j \rfloor} (1_{[0,y)}(\{T^k(x) - u\}) - y) - \rho \right)^2 dy\,dz$$

$$\geq 2^{-8} j \log_2 q.$$

Bearing in mind that $j \geq n - [2\log_q n] - 1$ and

$$4q^2(n - j + 2)^2 \leq 4q^2(2\log_q n + 3)^2 \leq 2^{-10} j \log_2 q,$$

for $n \geq n_0(q)$, we obtain the assertion of the lemma from (2.78).   ∎

LEMMA 2.14: *Let* $\mu \in [1, 3]$, $n \geq j \geq j_0 = n - [2\log_q n + 1]$, *and*

$$(2.79) \qquad f_0^{(\mu)}(M, x, y, u) = \sum_{i=0}^{d_2} \zeta_i^{(\mu)}(M, x, y, u).$$

Then

$$(2.80) \qquad 2^{10} q^4 n \geq D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) \geq 0.7 \cdot 2^{-11} n \log_2 q$$

for $n \geq n_1(q)$.

Proof:

$$(2.81) \qquad \zeta_{-1}(M, y, u) = \sum_{i=0}^{d_2} \sum_{\nu\in\Delta_i'} \xi_\nu(M, x, y, u),$$

where

$$(2.82) \qquad \Delta_0' = [0, k_0], \quad \Delta_i' = [id_1 - 2k_0, id_1 + k_0] \quad \text{for } 1 \leq i \leq d_2,$$

$$(2.83) \qquad d_3 = \min(d_1 d + k_0, n),$$

and

$$(2.84) \qquad \Delta'_{d_2} = [d_1 d_2 - 2k_0, d_3] \cup (\max(d_3, j_0 - 2k_0), n];$$

$$(2.85) \qquad \zeta_{-2}^{(\mu)}(M, x, y, u) = \sum_{i=0}^{d_2} \sum_{\nu \in \Delta_i} (\xi_\nu(M, x, y, u) - \xi_{\nu, k_0}^{(\mu)}(M, x, y, u)).$$

It follows from (2.25), (2.66)–(2.68), (2.79) and (2.81)–(2.85) that

$$(2.86) \quad f(M, x, y - u, u) = f_0^{(\mu)}(M, x, y, u) + \zeta_{-1}(M, x, y, u) + \zeta_{-2}^{(\mu)}(M, x, y, u).$$

By (2.81)–(2.84), we see that

$$(2.87) \qquad |\zeta_{-1}(M, x, y, u)| \leq 2q(d_2 + 1)(6k_0 + 2) \leq 2^5 q k_0 d_2.$$

Applying Lemma 2.9, we obtain for $\nu \leq j_0 - k_0$

$$(2.88) \qquad D_j^{(\mu)}(\xi_{\nu, k_0}^{(\mu)} - \xi_\nu) \leq 4q E_j^{(\mu)}|\xi_{\nu, k_0}^{(\mu)} - \xi_\nu| \leq 60q^{2-k_0} \leq 60q^2/n^2.$$

Now by (2.67) and (2.85), we conclude that

$$(2.89) \qquad D_j^{(\mu)}(\zeta_{-2}^{(\mu)}) \leq n^2 \max_\nu D_j^{(\mu)}(\xi_{\nu, k_0}^{(\mu)} - \xi_\nu) \leq 60q^2.$$

Using (2.78) and (2.86)–(2.89), we get

$$\begin{aligned} D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) \geq &\frac{1}{2} D_j^{(\mu)}(f(M, x, y - u, u)) - 2D_j^{(\mu)}(\xi_{-1}) \\ &- 2D_j^{(\mu)}(\zeta_{-2}^{(\mu)}) \geq \frac{1}{2} D_j^{(\mu)}(f) - 2^{11} q^2 k_0^2 d_2^2 - 120q^2. \end{aligned}$$

Then, by Lemma 2.13,

$$\begin{aligned} D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) \geq &2^{-11}(n - [2\log_q n + 1]) \log_2 q - 120q^2 \\ &- 2^{11} q^2 n^{2/3} (2\log_q n + 1)^2. \end{aligned}$$

Hence there exists $n_1 = n_1(q)$ such that

$$D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) \geq 0.7 \cdot 2^{-11} n \log_2 q \quad \text{for } n \geq n_1.$$

The right hand side of (2.79) is proved.

By Lemma 2.6 and (2.66)–(2.68), we have that $\zeta_i^{(\mu)}$ $(0 \leq i \leq d_2)$ are independent random variables.

In view of (2.79), we see that

$$D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) = \sum_{i=0}^{d_2} D_j^{(\mu)}(\zeta_i^{(\mu)}) = \sum_{i=0}^{d_2} D_j^{(\mu)}\left(\sum_{\nu \in \Delta_i} \xi_{\nu, k_0}^{(\mu)}\right).$$

From (2.67) and (2.68), we find that $\max_{i,\nu \in \Delta_i} \nu \le j_0 - 2k_0$.

Therefore, we can apply (2.64):

$$D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)) \le 2^{10} q^4 n.$$

This is the desired result.   ∎

We need the following generalization of the Berry–Essén theorem (see [H, p. 62]):

Suppose that for each variable k, $x_{k0}, x_{k1}, \ldots, x_{kk-1}$ are independent with zero means, $\delta \in (0, 1]$,

$$(2.90) \qquad \sum_{i=0}^{k-1} E(x_{ki}^2) = 1, \quad k \ge 1,$$

$$(2.91) \qquad \max_{0 \le i \le k-1} E(x_{ki}^2) \to 0 \quad \text{as } k \to \infty.$$

Then there exists a positive universal constant C such that

$$(2.92) \qquad \left| P\left(\sum_{i=0}^{k-1} x_{ki} < t \right) - \Phi(t) \right| < \frac{C}{1 + x^2} \sum_{i=0}^{k-1} E(|x_{ki}|^{2+\delta}).$$

LEMMA 2.15: *Let $\mu \in [1, 3]$, and $n \ge j \ge j_0$. Then*

$$(2.93) \qquad \left| \frac{1}{q^j} \sum_{M=0}^{q^j-1} \int_0^1 I\left(\frac{\hat{f}_0^{(\mu)}(M + (N_j + d_j)q^j, x, y, u)}{(D_{j,(N_j+d_j)q^j}^{(\mu)}(f_0^{(\mu)}(M, x, y, u)))^{1/2}} < t \right) dy - \Phi(t) \right|$$
$$\le C 2^{28} q^3 n^{-1/6} \log_q^{3/2} n, \quad n \ge n_1(q).$$

Proof: We use Berry–Essén's inequality (2.92) with $\delta = 1$, $k = d_2 + 1$, and $x_{k\nu} = \hat{\zeta}_\nu^{(\mu)}/d_{0,\mu}$, with $(d_{0,\mu})^2 = D_j^{(\mu)}(f_0(M, x, y, u))$, $0 \le \nu \le d_2$. It follows from (2.66)–(2.68) and Lemma 2.6 that the variables $\hat{\zeta}_0^{(\mu)}, \hat{\zeta}_1^{(\mu)}, \ldots, \hat{\zeta}_{d_2}^{(\mu)}$ are independent with zero means.

In view of (2.79), we conclude that (2.90) holds. Using (2.65), (2.66), Lemma 2.11 and Lemma 2.14, we get that

$$E_j^{(\mu)}(\hat{\zeta}_\nu^{(\mu)}/d_{0,\mu})^2 = D_j^{(\mu)}(\zeta_\nu^{(\mu)}/d_{0,\mu})$$
$$\le 2^{10} q^4 d_1 (D_j^{(\mu)}(f_0^{(\mu)}(M, x, y, u)))^{-1} = O(n^{-1/3}),$$

and (2.91) holds.

Bearing in mind Ljapanov's inequality

$$E_j^{(\mu)}|\omega|^3 \le (E_j^{(\mu)}\omega^4)^{3/4},$$

we obtain from Lemma 2.12 and Lemma 2.14

$$E_j^{(\mu)}|\zeta_\nu^{(\mu)}/d_{0,\mu}|^3 \leq (2^9 q^4 d_1^2 (2k_0+1)^2 \cdot (2^{-12} n \log_2 q)^2)^{3/4}$$
$$\leq (2^{36} q^4 n^{-2/3} \log_q^2 n)^{3/4} = 2^{27} q^3 n^{-1/2} \log_q^{3/2} n.$$

Now, by (2.65) and (2.92), the assertion of the lemma follows. ∎

3. Proofs of theorems

The three theorems will be proved together in the same way. We will use the index $\mu = 1, 2, 3$ in each case corresponding to Theorem 1.1, Theorem 1.2 and Theorem 1.3. Let

$$(3.1) \qquad N = \sum_{i=0}^n c_i q^i, \quad c_n \neq 0, \quad c_i \in \{0, \ldots, q-1\},$$

$$(3.2) \qquad E^{(\mu)} h(M, x, y, u) = E^{(\mu)} h = \frac{1}{N} \sum_{M=0}^{N-1} \int_{G_\mu} h(M, x, y, u) dw_\mu,$$

$$(3.3) \qquad D^{(\mu)}(h(M, x, y, u)) = D^{(\mu)}(h) = E^{(\mu)} (h - E^{(\mu)} h)^2,$$

$$(3.4) \qquad A^{(\mu)}(t) = \frac{1}{N} \sum_{M=0}^{N-1} \int_{G_\mu} I\Big(\frac{f(M, x, y, u) - E^{(\mu)} f(M, x, y, u)}{(D^{(\mu)} f(M, x, y, u))^{1/2}} < t\Big) dw_\mu,$$

with $\mu = 1, 2, 3$.

By (2.24)–(2.26) and (2.46), we see that

$$E^{(\mu)} f(M, x, y, u) = 0, \quad \text{for } \mu = 2, 3.$$

Using Fubini's theorem, (2.24) and (1.5), we obtain

$$E^{(\mu)} f(M, x, y, u) = \int_0^1 \int_{G_\mu} \sum_{k=0}^{\lfloor zN \rfloor} (1_{[u,u+y)}(T^k(x)) - y) dz dw_\mu.$$

By (3.3), (1.8), (1.11) and (1.13), we find that

$$E^{(1)} f(M, x, y, u) = e(x, u, N),$$

and

$$(3.5) \qquad D^{(\mu)}(f(M, x, y, u)) = \sigma_\mu \log_q N, \quad \text{for } \mu = 1, 2, 3.$$

Now, using Fubini's theorem, we obtain from (3.4) that the left sides of (1.7), (1.10) and (1.12) are equal, respectively, to

$$|A^{(\mu)}(t) - \Phi(t)| \quad \text{for } \mu = 1, 2, 3.$$

Let

(3.6)

$$A_{N,j,d_j}^{(\mu)} = \frac{1}{q^j} \sum_{M=0}^{q^j-1} \int_{G_\mu} 1$$
$$\times I\Big(\frac{f(M + (N_j + d_j)q^j, x, y, u) - E^{(\mu)} f(M, x, y, u)}{(D^{(\mu)}(f(M, x, y, u))^{1/2}} < t) \Big) dw_\mu,$$

where

$$N_j = q[N/q^{j+1}], \quad j = 0, 1, \ldots, n.$$

It is easy to verify that

$$A^{(\mu)}(t) = \sum_{j=0}^{n} \sum_{d_j=0}^{c_j-1} \frac{q^j}{N} A_{N,j,d_j}^{(\mu)}(t).$$

We see that for $j_0 = n - [2\log_q n + 1))$,

$$\left| A^{(\mu)}(t) - \Phi(t) - \sum_{j=j_0+1}^{n} \sum_{d_j=0}^{c_j-1} \frac{q^j}{N} (A_{N,j,d_j}^{(\mu)}(t) - \Phi(t)) \right|$$
$$\leq \sum_{j=0}^{j_0} \sum_{d_j=0}^{c_j-1} \frac{q^j}{N} |A_{N,j,d_j}^{(\mu)}(t) - \Phi(t)| \leq \frac{2q^{j_0+2}}{N} \leq \frac{2q^3}{n^2}.$$

Hence

$$|A^{(\mu)}(t) - \Phi(t)| \leq \frac{2q^3}{n^2} + \sum_{j=j_0+1}^{n} \sum_{d_j=0}^{c_j-1} \frac{q^j}{N} |A_{N,j,d_j}^{(\mu)}(t) - \Phi(t)|$$
$$\leq \frac{2q^3}{n^2} + q \max_{j \in (j_0, n]} \max_{d_j \in [0, c_j)} |A_{N,j,d_j}^{(\mu)}(t) - \Phi(t)|.$$

Thus to prove Theorem 1.1–Theorem 1.3, it is sufficient to verify that

(3.7) $$D^{(\mu)}(f(M, x, y, u)) \in [2^{-12}\log_2 q, 2^{12}q^4]$$

and

(3.8) $$A_{N,j,d_j}^{(\mu)}(t) - \Phi(t) = O(n^{-1/7})$$

for all $t \in \mathbb{R}$, $j \in (j_0, n]$, and $d_j \in [0, c_j)$ where O constant depends only on q.

We will conclude (3.7) and (3.8) from Lemma 2.14 and Lemma 2.15. To this end we need the following estimate of expectations:

LEMMA 3.1: *Let $\mu \in [1,3]$, and $n \geq j > j_0 = n - k_0$, $k_0 = [2 \log_q n] + 1$. Then*

$$(3.9) \qquad E^{(\mu)} f(M, x, y - u, u) = E^{(\mu)}_{j,(N_j + d_j)q^j} f_0^{(\mu)}(M, x, y, u) + O(n^{1/3} \log_q n)$$

and

$$(3.10) \quad D^{(\mu)} f(M, x, y - u, u) = D^{(\mu)}_{j,(N_j + d_j)q^j} f_0^{(\mu)}(M, x, y, u) + O(n^{5/6} \log_q n).$$

Proof: From Lemma 2.6, (2.79), (2.85) and (2.66)–(2.68), we find that $f_0^{(\mu)}(M + (N_\nu + b_\nu)q^\nu, x, y, u)$ and $\zeta_{-2}^{(\mu)}(M + (N_\nu + b_\nu)q^\nu, x, y, u)$ depend only on the first $n - k_0 = j_0$ digits of the q-expression of $M + (N_\nu + b_\nu)q^\nu$. Hence for $\nu, j > j_0$

$$(3.11) \qquad E^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} f_0^{(\mu)}(M, x, y, u) = E^{(\mu)}_{j,(N_j + d_j)q^j} f_0^{(\mu)}(M, x, y, u),$$

$$(3.12) \qquad D^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} (f_0^{(\mu)}(M, x, y, u)) = D^{(\mu)}_{j,(N_j + d_j)q^j} (f_0^{(\mu)}(M, x, y, u)),$$

$$(3.13) \qquad E^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} |\zeta_{-2}^{(\mu)}| = E^{(\mu)}_{j,(N_j + d_j)q^j} |\zeta_{-2}^{(\mu)}|,$$

and

$$(3.14) \qquad D^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} (\zeta_{-2}^{(\mu)}) = D^{(\mu)}_{j,(N_j + d_j)q^j} (\zeta_{-2}^{(\mu)}).$$

From (2.49), (2.67) and (2.85), we conclude that

$$(3.15) \qquad E^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} |\zeta_{-2}^{(\mu)}| \leq 15 q^2 n / q^{k_0} \leq 15 q^2 / n.$$

In view of (3.1), (3.2), (2.44) and (2.45), we have that

$$(3.16) \qquad E^{(\mu)} f(M, x, y - u, u) = \sum_{\nu=0}^{n} \sum_{b_\nu = 0}^{c_\nu - 1} \frac{q^\nu}{N} E^{(\mu)}_{\nu,(N_\nu + b_\nu)q^\nu} f(M, x, y - u, u)$$

and
(3.17)

$$E^{(\mu)}_{j,(N_j + d_j)q^j} f_0^{(\mu)}(M, x, y - u, u) = \sum_{\nu=0}^{n} \sum_{b_\nu = 0}^{c_\nu - 1} \frac{q^\nu}{N} E^{(\mu)}_{j,(N_j + d_j)q^j} f_0^{(\mu)}(M, x, y - u, u).$$

It follows from (2.25) and (2.79) that

$$(3.18) \qquad |f(M, x, y, u)| \leq 2q(n + 1) \quad \text{and} \quad |f_0(M, x, y, u)| \leq 2qn.$$

Hence

$$
|E^{(\mu)} f(M,x,y-u,u) - E^{(\mu)}_{j,(N_j+d_j)q^j} f_0^{(\mu)}(M,x,y-u,u)|
$$

$$
\leq 8qn \sum_{\nu=0}^{j_0} \frac{q^\nu}{N} + \sum_{\nu=j_0+1}^{n} \sum_{b_\nu=0}^{c_\nu-1} \frac{q^\nu}{N} |E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} f^{(\mu)}(M,x,y-u,u)
$$

(3.19)
$$
- E^{(\mu)}_{j,(N_j+d_j)q^j} f_0^{(\mu)}(M,x,y,u)|
$$

$$
\leq 8nq^{2-k_0} + q^2 \max_{\nu\in(j_0,n)} \max_{b_\nu\in[0,c_\nu)} |E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} f(M,x,y-u,u)
$$

$$
- E^{(\mu)}_{j,(N_j+d_j)q^j} f_0^{(\mu)}(M,x,y,u)|.
$$

By (3.11), (3.15), (2.86), (2.87), (2.62) and (2.65), we see that

$$
|E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} f(M,x,y-u,u) - E^{(\mu)}_{j,(N_j+d_j)q^j} f_0^{(\mu)}(M,x,y,u)|
$$

(3.20)
$$
= |E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} (f(M,x,y-u,u) - f_0^{(\mu)}(M,x,y,u))|
$$

$$
\leq E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} |\zeta_{-1}(M,x,y,u)| + E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} |\zeta_{-2}^{(\mu)}(M,x,y,u)|
$$

$$
\leq 2^5 q k_0 d_2 + 15 q^2/n = O(n^{1/3} \log_q n), \quad \text{for } \nu, j > j_0.
$$

Using (3.19), we obtain the assertion (3.9).

Now consider (3.10): Similarly to (3.17) and (3.18), we find from (2.45)–(2.48), (3.2) and (3.3) that

$$
D^{(\mu)}(f(M,x,y,u)) = \sum_{\nu=0}^{n} \sum_{b_\nu=0}^{c_\nu-1} \frac{q^\nu}{N} E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} (f(M,x,y,u) - E^{(\mu)} f(M,x,y,u))^2
$$

and

$$
D^{(\mu)}_{j,(N_j+b_j)q^j} (f_0^{(\mu)}(M,x,y,u)) = \sum_{\nu=0}^{n} \sum_{b_\nu=0}^{c_\nu-1} \frac{q^\nu}{N} D^{(\mu)}_{j,(N_j+b_j)q^j} (f_0^{(\mu)}(M,x,y,u)).
$$

Using (3.18) we conclude analogously to (3.19) that

$$
|D^{(\mu)}(f(M,x,y,u)) - D^{(\mu)}_{j,(N_j+b_j)q^j} (f_0^{(\mu)}(M,x,y,u))|
$$

$$
\leq 16q^2 n^2 \sum_{\nu=0}^{j_0} \frac{q^\nu}{N}
$$

$$
+ \sum_{\nu=j_0+1}^{n} \sum_{b_\nu=0}^{c_\nu-1} \frac{q^\nu}{N} |E^{(\mu)}_{\nu,(N_\nu+b_\nu)q^\nu} (f(M,x,y,u) - E^{(\mu)} f(M,x,y,u))^2
$$

(3.21)
$$
- D_{j,(N_j+b_j)q^j} (f_0^{(\mu)}(M,x,y,u))|
$$

$$\leq 16q^3$$
$$+ q^2 \max_{\nu \in (j_0, n]} \max_{b_\nu \in [0, c_\nu]} |E^{(\mu)}_{\nu, (N_\nu + b_\nu) q^\nu} (f(M, x, y, u) - E^{(\mu)} f(M, x, y, u))^2$$
$$- D^{(\mu)}_{j, (N_j + d_j) q^j} (f^{(\mu)}_0 (M, x, y, u))|.$$

It is easy to see that

$$(f - E^{(\mu)} f)^2 = (f - E^{(\mu)}_\nu f + (E^{(\mu)} - E^{(\mu)}_\nu) f)^2$$
$$+ 2(f - E^{(\mu)}_\nu f)((E^{(\mu)} - E^{(\mu)}_\nu) f) + ((E^{(\mu)} - E^{(\mu)}_\nu) f)^2$$

and

$$E^\mu_\nu (f - E^{(\mu)} f)^2 = D^{(\mu)}_\nu (f) + ((E^{(\mu)} - E^{(\mu)}_\nu) f)^2.$$

Applying (3.9) to $\nu = j > j_0$ and $b_\nu = d_\nu$, we obtain

$$E^{(\mu)}_{\nu, (N_\nu + b_\nu) q^\nu} (f - E^{(\mu)} f)^2 = D^{(\mu)}_{\nu, (N_\nu + b_\nu) q^\nu} (f) + O(n^{2/3} \log^2_q n).$$

Using relations (3.21) and (3.12), we conclude that to prove (3.10) it is sufficient to verify that

$$(3.22) \qquad D^{(\mu)}_{\nu, (N_\nu + b_\nu) q^\nu} (f) = D^{(\mu)}_{\nu, (N_\nu + b_\nu) q^\nu} (f^{(\mu)}_0) + O(n^{5/6} \log_q n)$$

for $\nu \in (j_0, n]$.

By (2.86) we find that

$$D^{(\mu)}_\nu (f) = E^{(\mu)}_\nu ((f^{(\mu)}_0 - E^{(\mu)}_\nu f_0) + (\zeta_{-1} + \zeta^{(\mu)}_{-2} - E^{(\mu)}_\nu (\zeta_{-1} + \zeta^{(\mu)}_{-2})))^2$$
$$(3.23) \qquad = D^{(\mu)}_\nu (f^{(\mu)}_0) + D^{(\mu)}_\nu (\zeta_{-1} + \zeta^{(\mu)}_{-2})$$
$$+ 2E^{(\mu)}_\nu (f^{(\mu)}_0 - E^{(\mu)}_\nu f^{(\mu)}_0)(\zeta_{-1} + \zeta^{(\mu)}_{-2} - E^{(\mu)}_\nu (\zeta_{-1} + \zeta^{(\mu)}_{-2})).$$

In view of (2.65), (2.87), (2.89) and (3.14), we have that

$$D^{(\mu)}_\nu (\zeta^{(\mu)}_{-2}) \leq 16q^2 \quad \text{and} \quad D^{(\mu)}_\nu (\zeta_{-1}) \leq 2^{16} q^2 n^{2/3} \log^{2/3}_q n.$$

Hence

$$(3.24) \qquad D^{(\mu)}_\nu (\zeta_{-1} + \zeta^{(\mu)}_{-2}) = O(n^{2/3} \log^{2/3}_q n).$$

Bearing in mind that $|f^{(\mu)}_0 (M, x, y, u)| \leq 2qn$, from (3.20) we obtain

$$(3.25) \qquad E^{(\mu)}_\nu (|f^{(\mu)}_0 - E^{(\mu)}_\nu f^{(\mu)}_0| \cdot |\zeta^{(\mu)}_{-2} - E^{(\mu)}_\nu \zeta^{(\mu)}_{-2}|) \leq 2qn E^{(\mu)}_\nu |\zeta^{(\mu)}_{-2} - E^{(\mu)}_\nu \zeta^{(\mu)}_{-2}|$$
$$\leq 30q^3.$$

Now, from Lemma 3.1, (3.12), (2.87) and (2.65) we see that

(3.26)
$$E_\nu^{(\mu)}(|f_0^{(\mu)} - E_\nu^{(\mu)} f_0^{(\mu)}| \cdot |\zeta_{-1} - E_\nu^{(\mu)} \zeta_{-1}|) \le 2^5 q k_0 d_2 E_\nu^{(\mu)} |f_0^{(\mu)} - E_\nu^{(\mu)} f_0^{(\mu)}|$$
$$\le 2^8 q n^{1/3} \log_q n (D_\nu^{(\mu)}(f_0^{(\mu)}))^{1/2} \le 2^{13} q^3 n^{5/6} \log_q n.$$

Substituting (3.24)–(3.26) into (3.23), we obtain the assertion (3.22). The lemma is proved. ∎

Completion of the Proof of Theorem 1.1–Theorem 1.3: Let $j > j_0$. We see that

(3.27)
$$\frac{f - E^{(\mu)} f}{\sqrt{D^{(\mu)}(f)}} - \frac{f_0^{(\mu)} - E_j^{(\mu)} f_0^{(\mu)}}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} = \frac{f - f_0^{(\mu)}}{\sqrt{D^{(\mu)}(f)}} + \frac{E_j^{(\mu)} f_0^{(\mu)} - E^{(\mu)} f}{\sqrt{D^{(\mu)}(f)}}$$
$$+ (f_0^{(\mu)} - E_j^{(\mu)} f_0^{(\mu)}) \left(\frac{1}{\sqrt{D^{(\mu)}(f)}} - \frac{1}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} \right)$$
$$:= A_1 + A_2 + A_3.$$

It follows from Lemma 2.14 and Lemma 3.1 that

(3.28)
$$A_2 = O(n^{-1/6} \log_q n)$$

and

(3.29)
$$\frac{1}{\sqrt{D^{(\mu)}(f)}} - \frac{1}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} = \frac{D_j^{(\mu)}(f_0^{(\mu)}) - D^{(\mu)}(f)}{\sqrt{D^{(\mu)}(f)} \sqrt{D_j^{(\mu)}(f_0^{(\mu)})} (\sqrt{D^{(\mu)}(f)} + \sqrt{D_j^{(\mu)}(f_0^{(\mu)})})}$$
$$= O(n^{-2/3} \log_q n).$$

Let

(3.30)
$$\Omega_1^{(\mu)} = \Big\{ (w_\mu, M) \in G_\mu \times [(N_j + d_j)q^j, (N_j + d_j + 1)q^j) \big|$$
$$|f_0^{(\mu)}(M, x, y, u) - E_j^{(\mu)} f_0^{(\mu)}(M, x, y, u)| > \sqrt{D_j^{(\mu)} f_0^{(\mu)}} \Big\}.$$

By Tchebishev's inequality and Lemma 2.14,

(3.31)
$$\frac{1}{q^j} \operatorname{mes} \Omega_1^{(\mu)} \le \frac{1}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} = O(n^{-1/2}).$$

In view of Lemma 2.14, (3.29) and (3.30), we have that

(3.32)
$$A_3 = O\big(n^{-2/3} \log_q n \sqrt{D_j^{(\mu)}(f_0^{(\mu)})}\big) = O(n^{-1/6} \log_q n)$$
$$\text{for } (w_\mu, M) \notin \Omega_1^{(\mu)}.$$

Now consider A_1: Let

$$(3.33) \quad \Omega_2^{(\mu)} = \left\{ (w_\mu, M) \in G_\mu \times [(N_j + d_j)q^j, (N_j + d_j + 1)q^j) \big| \zeta_{-2}^{(\mu)} \notin \left[0, \frac{n}{q^{k_0-1}} \right) \right\}.$$

From (2.86), (2.87) and (2.62), we find that

$$|f - f_0^{(\mu)}| = |\zeta_{-1} + \zeta_0^{(\mu)}| \leq 2^5 q k_0 d_2 + 1 \quad \text{for } (w_\mu, M) \notin \Omega_2^{(\mu)}.$$

By Lemma 2.14 and (2.65)

$$(3.34) \qquad\qquad A_1 = O(n^{-1/6} \log_q n) \quad \text{for } (w_\mu, M) \notin \Omega_2^{(\mu)}.$$

Using (2.85), we have that

$$\Omega_2^{(\mu)} \subset \bigcup_{i=0}^{d_2} \bigcup_{\nu \in \Delta_i} \Omega_{2,\nu}^{(\mu)},$$

where

$$\Omega_{2,\nu}^{(\mu)} = \Big\{ (w_\mu, M) \in G_\mu \times [(N_j + d_j)q^j, (N_j + d_j + 1)q^j) \big|$$
$$\xi_\nu - \xi_{\nu,k_0}^{(\mu)} \notin \left(-\frac{1}{q^{k_0-1}}, \frac{1}{q^{k_0-1}} \right) \Big\}.$$

Applying Lemma 2.7, Lemma 2.8 and (2.27)–(2.30), we show that

$$\frac{1}{q^j} \operatorname{mes} \Omega_{2,\nu}^{(\mu)}$$

$$\leq 2q \sup_{x \in [0,1)} \max_{L \in B_1} \operatorname{mes} \left\{ y \in [0,1) \big| g_{\nu,k_0}(L, x, y) - g_\nu(L, x, y) \right\} \notin \left[0, \frac{1}{q^{k_0}} \right) \right\}$$

$$+ q \sup_{x,y \in [0,1)} \max_{a \in B_2} \frac{1}{q^{k_0}} \sum_{L=0}^{q^{k_0}-1} I(g_{\nu,k_0}^{(1)}(L + aq^{k_0}, x, y) - g_\nu(L + aq^{k_0}, x, y))$$

$$\leq 5/q^{k_0-1},$$

where

$$B_1 = [(N - j + d_j)q^j, \ (N_j + d_j + 1)q^j)$$

and

$$B_2 = [(N_j + d_j)q^{j-k_0}, (N_j + d_j + 1)q^{j-k_0}).$$

Hence

$$(3.35) \qquad\qquad \frac{1}{q^j} \operatorname{mes} \Omega_2^{(\mu)} \leq 5n/q^{k_0-1} \leq 5q/n.$$

We denote the left side of (3.27) by Δ. Let

$$\Delta_1 = \inf_{(w_\mu, M) \notin \Omega_1^{(\mu)} \cup \Omega_2^{(\mu)}} \Delta \quad \text{and} \quad \Delta_2 = \sup_{(w_\mu, M) \notin \Omega_1^{(\mu)} \cup \Omega_2^{(\mu)}} \Delta.$$

From (3.28), (3.32) and (3.34), we conclude that

$$(3.36) \qquad\qquad \Delta_1, \Delta_2 = O(n^{-1/6} \log_q n).$$

We find that for $(w_\mu, M) \in \Omega_1^{(\mu)} \cup \Omega_2^{(\mu)}$,

$$I\left(\frac{f_0^{(\mu)}(M, x, y, u) - E_j^{(\mu)} f_0^{(\mu)}}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} < t - \Delta_2 \right)$$

$$\leq I\left(\frac{f(M, x, y - u, u) - E^{(\mu)} f}{\sqrt{D^{(\mu)} f}} < t \right)$$

$$\leq I\left(\frac{f_0^{(\mu)}(M, x, y, u) - E_j^{(\mu)} f_0^{(\mu)}}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} < t - \Delta_1 \right).$$

According to (2.24), the function $f(M, x, y, u)$ is periodic with period 1 over y. Hence, by (3.6), we see that

$$\frac{1}{q^j} \sum_{M=0}^{q^j - 1} \int_{G_\mu} I\left(\frac{f_0^{(\mu)}(M + (N_j + d_j)q^j, x, y, u) - E_j^{(\mu)} f_0^{(\mu)}}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} < t - \Delta_2 \right) dw_\mu$$

$$- \frac{1}{q^j} \operatorname{mes}(\Omega_1^{(\mu)} \cup \Omega_2^{(\mu)}) \leq A_{N,j,d_j}^{(\mu)}(t) \leq \frac{1}{q^j} \operatorname{mes}(\Omega_1^{(\mu)} \cup \Omega_2^{(\mu)})$$

$$+ \frac{1}{q^j} \sum_{M=0}^{q^j - 1} 1 \int_{G_\mu} I\left(\frac{f_0^{(\mu)}(M + (N_j + d_j)q^j, x, y, u) - E_j^{(\mu)} f_0^{(\mu)}}{\sqrt{D_j^{(\mu)}(f_0^{(\mu)})}} < t - \Delta_1 \right) dw_\mu.$$

Applying Lemma 2.15, we obtain

$$A_{N,j,d_j}^{(\mu)}(t) - \Phi(t) = O\left(n^{-1/6} \log_q^{3/2} n + \frac{1}{q^j} \operatorname{mes}(\Omega_1^{(\mu)} \cup \Omega_2^{(\mu)}) \right.$$

$$\left. + \max(|\Phi(t) - \Phi(t - \Delta_1)|, |\Phi(t) - \Phi(t - \Delta_2)|) \right),$$

where O constant depends only on q. Since

$$|\Phi(t) - \Phi(t + v)| = \frac{1}{\sqrt{2\pi}} \left| \int_t^{t+v} e^{-s^2/2} ds \right| < |v|,$$

from (3.31), (3.35) and (3.36), we conclude that

$$A^{(\mu)}_{N_j,j,d_j}(t) - \Phi(t) = O(n^{-1/6}(\log_q n)^{3/2}),$$

where O constant depends only on q. The assertion (3.8) is proved.

Now from (3.5), Lemma 2.14 and (3.10), we obtain (3.7) and the assertion of Theorem 1.1–Theorem 1.3.   ∎

4. Appendix

Here we consider a slight modification of the proof (see [KN, pp. 100–104]) of Roth's theorem.

Let $\beta_1,\ldots,\beta_N$ be given points in the s-dimensional unit cube $[0,1)^s$, $\beta_i = (\beta_{i1},\ldots,\beta_{is})$, $i = 1,\ldots,N$; and let $A(x)$ denote the number of points β_i, $1 \le i \le N$, in the box $[0,x_1) \times \cdots \times [0,x_s)$, with $x = (x_1,\ldots,x_s)$.

We use notation of [KN, pp. 100–104]. From [KN, Lemma 2.1, p. 100 and p. 104], we see that

$$(4.1) \qquad \int_{[0,1)^s} F(x)dx = 0 \quad \text{and} \quad \int_{[0,1)^s} A(x)F(x)dx = 0.$$

By the Cauchy–Schwartz inequality, [KN, p. 104] and (4.1), we obtain

$$\int_{[0,1)^s} (A(x) - Nx_1\ldots x - \rho)^2 dx$$

$$\ge \left(\int_{[0,1)^s} (A(x) - Nx_1\ldots x_s - \rho)F(x)dx \right)^2 \left(\int_{[0,1)^s} F^2(x)dx \right)^{-1}$$

$$= \left(\int_{[0,1)^s} (A(x) - Nx_1\ldots x_x)F(x)dx \right)^2 \times \left(\int_{[0,1)^s} F^2(x)dx \right)^{-1}$$

$$\ge 2^{-8s}(s-1)^{1-s}(\log_2 N)^{s-1}.$$

Hence

$$(4.2) \qquad \inf_{\rho \in \mathbb{R}} \int_{[0,)^s} (A(x) - Nx_1\ldots x_s - \rho)^2 dx \ge 2^{-8s}(s-1)^{1-s}(\log_2 N)^{s-1}.$$

We apply this inequality to $s = 2$, $x_1 = y$, $x_2 = z$, $\beta_{i1} = \beta_{i-1}$, and $\beta_{i2} = (i-1)/N$, $1 \le i \le N$. It is easy to see that $(i-1)/N < z \Leftrightarrow i-1 < \lfloor zN \rfloor$. Hence $A(y,z) = \sum_{i=0}^{\lfloor zN \rfloor} 1_{[0,y)}(\beta_i)$. Using (4.2) we find that

$$\inf_{\rho \in \mathbb{R}} \int_0^1 \int_0^1 \left(\sum_{i=0}^{\lfloor zN \rfloor} (1_{[0,y)}(\beta_i) - y) - \rho \right)^2 dydz \ge 2^{-8}\log_2 N. \quad ∎$$

References

[B1]   J. Beck, *Randomness of $n\sqrt{2}$ mod 1 and a Ramsey property of the hyperbola*, in *Sets, Graphs and Numbers (Budapest, 1991)*, Colloquia Mathematica Societatis János Bolyai, 60 (G. Halász, L. Lovász, D. Miklós and I. Szöny, eds.), North-Holland, Amsterdam, 1992, pp. 23–66.

[B2]   J. Beck, *Diophantine approximation and quadratic fields*, in *Number Theory (Egez, 1996)* (K. Györy, A. Pethö and V. Sós, eds.), de Gruyter, Berlin, 1998, pp. 55–93.

[B3]   J. Beck, *From probabilistic Diophantine approximation to quadratic fields*, in *Random and Quasi-Random Points Sets* (P. Kellekalek and G. Larcher, eds.), Lecture Notes in Statistics, Vol. 138, Springer, New York, 1998, pp. 1–48.

[Be]   S. N. Bernstein, *Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes*, Mathematische Annalen **97** (1926), 1–59.

[Fr]   N. A. Friedman, *Introduction to Ergodic Theory,* Van Nostrand Reinhold Co., New York, 1970.

[H]   P. Hall, *Rates of Convergence in the Central Limit Theorem*, Pitman Advanced Publishing Program, Boston, 1982.

[KN]   L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[LP]   B. Lapeyre and G. Pages, *Families of sequences with low discrepancy obtained by iteration of transforms of [0,1]*, Comptes Rendus de l'Académie des Sciences, Paris, Série I **308** (1989), 507–509.

[L]   M. Levin, *Stochastic behavior of the ergodic adding machine,* in *Proceedings of the Annual Meeting of the Israel Mathematical Union*, Haifa University, Haifa, 2000, p. 41.

[P]   G. Pages, *Van der Corput sequences, Kakutani transforms and one-dimensional numerical integration*, Journal of Computational and Applied Mathematics **44** (1992), 21–39.

[Pe]   K. Petersen, *Ergodic Theory*, Cambridge University Press, 1983.

[PA]   P. D. Proinov and E. V. Atanasov, *On the distribution of the van der Corput generalized sequences*, Comptes Rendus de l'Académie des Sciences, Paris, Série I **307** (1988), 895–900.